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Asymmetric Exchanges

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Abstract

We emphasize the asymmetric character of exchanges between artists and scientists or mathematicians in the framework of multidisciplinary collaborations. As an example, we look at the perception of asymmetry in an artwork presented at the First European Asymmetry Symposium (FEAS), Nice, 15-16 March 2018, and its potential connection with the unsolved mathematical problem of maximal or asymptotically maximal asymmetric figures and distributions.

Keywords

arts; symmetry; asymmetry measures; chirality; most scalene triangle

I INTRODUCTION

As stated by Poey *et al* (2013), active involvement in arts can help learners in science and other fields explore different perspectives and internalize new ideas and ways of thinking. But sciences cannot be reduced to their experimental aspects: each time an equation is written to provide a model describing some natural phenomena, in physics, chemistry, biology, earth sciences, geography, econometrics, human sciences, etc., it means that the scientists are using mathematical tools. Thus mathematics cannot be dissociated from sciences. And indeed, there are strategies to integrate the arts in mathematic Dacey and Donovan (2013). Fenyvesi and Lähdesmäki (2017) noticed that both mathematics and arts are conceptual and symbolic 'languages' that humans have used in their attempts to depict their empirical perceptions and visions. Mathematical sculptors invoke mathematics not only for its elegant abstractions but also in hopes of speaking to the ways math underlies the world, often in hidden ways, as noticed by Omes (2017). In fact it was even recently asked if artists lead the way in mathematics: see Adams (2017).

Then it is impossible to miss the major role of symmetry in arts, science and mathematics. This is not a new topic. Shubnikov (1977) noticed that the perception of symmetry in art and in nature has been appreciated since antiquity, a remarkable example being Islamic art: see Rózsa (1986), Tennant (2008). This perception of symmetry is illustrated by a number of images in the books of Hahn (1989, 1998) and of Darvas (2007). Jablan (1995, 2002) showed the major role of symmetry in ornaments. Gerdes (2000) gave numerous examples of geometric creativity of handmade basket makers all around the world. In his book, Kappraff (2001) wrote that design science is a subject which may be considered to be a geometric bridge between art and science and devoted a whole chapter to symmetry. The role of symmetry has been investigated in literature, e.g. by Pavlović and Trinajstić (1986) and in poetry by Bonch-Osmolovskaya (2012), and in music by Banney (2015), while numerous other contributors could have been cited in these fields. Stylianou and Grzegorczyk (2005) noticed that familiarity with symmetry concepts allowed arts students to use abstract mathematical concepts as a tool in their art creation. Symmetry has also been tagged as connecting art and ethnography, linking geometric symmetry with interculturality: see Marchis (2009).

The relations between arts and beauty have been outlined through symmetry by Weyl (1952). It has been discussed by McManus (2005) and by Zaidel and Hessamian (2010). As noticed by Molnar and Molnar (1986), symmetry has always been considered an important factor in visual aesthetics. Quoting the French philosopher Montesquieu (1757) "*Une des principales causes des plaisirs de notre âme lorsqu'elle voit des objets c'est la facilité qu'elle a de les apercevoir et la raison qui fait que la symétrie plait à l'âme c'est qu'elle soulage*", which could be translated in English by "*One of the main causes of the pleasures of our soul when it sees objects is the ease with which they see them and the reason that symmetry pleases the soul is that it relieves*". However, in a recent study Weichselbaum *et al* (2018) showed that with higher art expertise, the ratings for the beauty of asymmetrical patterns significantly increased, while participants to the questionnaire still preferred symmetrical over asymmetrical patterns. But perceptual symmetry is not always identical to the symmetry defined by the mathematicians. This can be illustrated by the work Escher (1959), one of the most famous artists having produced artworks considered to be based on symmetry: see Schattschneider (2004).

The readers willing to read more about connections of arts with mathematics and sciences can find in annex A at the end of this paper a short list of journals and newsletters publishing papers in this field.

However, while many artists play with symmetry, not all of them think to symmetry as a way to exchange ideas with scientists and mathematicians. Furthermore, these exchanges appear to be asymmetric in the sense that what is sent may differ from what is received, while both parties get benefits from these exchanges. In the next section we present artworks illustrating such facts.

II FIRST EUROPEAN ASYMMETRY SYMPOSIUM (FEAS): ARTS-SCIENCES CROSSED PERSPECTIVES

2.1 FEAS, March 2018

Complex asymmetric systems such as the origin and evolution of asymmetric life, asymmetric amplification, asymmetric structures, asymmetry in economy and art – to name a few – are far from being understood and expressed and we expect that fundamental answers can be found by a transdisciplinary approach only that systematically complements knowledge of traditional individual disciplines. FEAS (2018) is the first European symposium of its kind. It was intended

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to attract leading international scientists and researchers and to advance our trans-disciplinary understanding in asymmetry-related phenomena. As noticed in the FEAS Editorial (2018), this symposium had a vigorously transdisciplinary programme, which aimed to focus and encourage research on asymmetry in systems as diverse as the mouse zygote and market economies, chemical structure, Japanese art and neuroscience. During FEAS Caroline Bouissou, artist at Villa Arson (Nice, France) met Michel Petitjean, researcher at University Paris Diderot. They began their exchange between art creation and science research. From there was the proposition of a mathematical view of art work directly linked to a mathematical specific research.

2.2 Crossed pespectives: arts-sciences

These crossed perspectives occurred between the coauthors of this paper, who both attended the FEAS symposium. Caroline Bouissou is a pluridisciplary artist attached to Villa Arson, an academic institute dedicated to contemporary art (`https://www.villa-arson.org/`). She uses different media as sound, installation, drawing, etc. (see more: `http://www.carolinebouissou.com`). She is part of the Asymmetry Group formed in University Côte d'Azur (UCA), Nice, France. Michel Petitjean is an academic researcher working at the University Paris Diderot, member of executive board of the International Symmetry Association, founded fifteen years ago: see ISA (2003).

For the FEAS Caroline Bouissou composed a purified installation shown in figure 1 with recognizable everyday life elements (glasses and glass surfaces). The perception of asymmetry in "*Reflections 2018*" and its potential connection with the unsolved mathematical problem of maximal or asymptotically maximal asymmetric figures and distributions constituted an echo to the research of Michel Petitjean. At first glance, "*Reflections 2018*" evokes mirror symmetry: there are glasses under glasses, separated by a glass (the mirror). Looking closer, it can be seen that there are true reflexions and false reflexions: glass is a matter distributed unsymmetrically around the mirror, so that the whole mass distribution is asymmetric. The installation is based on the contradiction of the symmetry desire provoked by a gap, a problem, an asymmetry. It is easier to identify a symmetric shape than an asymmetric one. Shapes corresponding one to another are more comfortable to get. This perception is perturbated at the exact moment when a shape that should correspond to another does not correspond anymore: rupture. All elements of the installation are transparent. Glasses and glass are all made of transparent glass and elements confuse with the glass surface. As with a water surface there is a reflexion but here this reflexion is materially represented: objects are the reflection of their identical copy. Thus the artist evokes asymmetry (see also annex B). As mentionned in the FEAS Editorial (2018): *More surprisingly, perhaps, asymmetry could be fundamental to aesthetics, too – perhaps reflecting what the art historian Martin Kemp calls a 'structural intuition' that lets us discern vitality in organic form; by contrast, geometric perfection creates a sense of sterility*."

In the next chapter we analyse what could mean "most asymmetric", in a mathematical sense. Some of the results are surprising, and it is concluded in section IV that the approach of asymmetry done by Caroline Bouissou was pertinent. In annex B are presented additional artworks which are not in the core of this study, but which are connected to it. We plan to develop in a future work a pluridisciplinary reading of these additional artworks.

2.3 Art-mathematics exchange

We present two researches stemming from two different fields which found here a meeting point. Terms, purposes and reasonings, methods and media of application differ from one to the other. In research-creation the gesture accompanies the thought and follow a chain of explorations with aesthetic, formal and abstract experiments from which the work of art is the result submitted to the projections, feelings and interpretations of visitors. The scientific research is based on experiments answering protocols and results into reports mostly addressed to specialists. Here there is an exchange more than a collaboration, it is one of the points the authors try to develop. Reading the work of art under a mathematical prism adds to it a new possible lecture and makes it the vehicle to communicate a mathematical research.

Figure 1: *Reflections 2018*. Glass and glasses: artwork designed by Caroline Bouissou.
Photographs: © Caroline Bouissou. On the top, version presented at FEAS (2018). On the bottom: a variant. Installations by Caroline Bouissou. The installation may vary, the elements and the dimensions can change, the principle is the same: reflections, false reflections. Some of the glasses are disposed symmetrically, some are disposed asymmetrically. They maintain the glass surface between them. Glasses reflect in glass surface. Real and false reflections get superimposed, producing an offset. Each element has many dimensions of symmetry, the artist plays with the title and the description of the installation itself: ”Reflection” being both the mirror reflection and the thought reflection. The objet and the surface of reflection have the same name ”glass”. Here the offset disposition of glasses is deeply moving the all symmetric composition into an asymmetric composition.

III MATHEMATICAL ASPECTS

3.1 Terminology

The word symmetry is usually applied to a large diversity of situations, not all of them being assumed to receive a mathematical meaning. A mathematical symmetry may be used as a way to modelize some physical situation where it is wished to recognize some symmetry. Weyl (1952) is often cited to have provided a mathematical definition of symmetry, while this latter is an intuitive definition rather than a mathematical one. Rosen (2008) considered that symmetry is "*immunity to a possible change*": this is an intuitive definition, too. We may retain the definition of Petitjean (2007), which is an unifying definition covering the cases of functions, geometric figures with or without colors, probability distributions, matrices, graphs, text strings, etc. Practically, all these cases can be handled through functions. The details are out of the scope of the present paper: the readers, including those who have only basic mathematical skills, are invited to look at the original publication of Petitjean (2007).

For clarity, we recall that two kinds of symmetry exist in the Euclidean space: direct symmetry and indirect symmetry. Roughly speaking, an object has a direct symmetry when it is recognized to be identical to itself after having been submitted to some combination of translations and rotations, the net result of this combination being assumed to be non null (i.e. the null result is the case of neither translation nor rotation). Still roughly speaking, an object has indirect symmetry when a mirror image of this object is recognized to be identical to the original object. When the the object has no indirect symmetry, it is declared to be chiral, and when it has indirect symmetry, it is declared to be achiral: this is what stated Lord Kelvin (1904). It should be mentioned that an older analysis of mirror symmetry was done more than two centuries ago by Kant (1768, 1770, 1783) while Kant's interpretation of mirror symmetry seems to be still discussed: see Hoefer (2000). In fact, it appears that chirality can be defined in any metric space, on not only in Euclidean spaces: see the full mathematical definition of Petitjean (2010); a more recent version of this paper is available in open access: see Petitjean (2018).

There are chiral objects which have direct symmetry, and there achiral objects (i.e. having indirect symmetry) which have no direct symmetry. Of course, there are objects having both types of symmetries together, such as a cube.

Asymmetry is defined as the full absence of symmetry, as stated by Saha and Chakraborty (2012), Hargittai and Hargittai (2009), and by Gal (2011). The word dissymetry seemed to be used for objects with a lack of some symmetry element, while in practice dissymetry appears to be an obsolete word for chirality: see Eliel (1997), Mislow (2002), Gal (2011), and Saha and Chakraborty (2012).

3.2 Measuring asymmetry

Stating that an object is symmetric or not refers to an ideal situation. An object can be nearly symmetric, thus arising the problem of how to measure quantitatively its degree of asymmetry. Many attempts to build such measures were performed in various areas of science since the end of 19^{th} century: see the review of Petitjean (2003). Most of these measures dealt with chirality and were shown to offer a serious lack of general applicability. The mathematical analysis of asymmetry measures being out of the scope of this present paper, we retain the following measures of asymmetry:

- DSI, the Direct Symmetry Index defined by Petitjean (1999), which is a scale and isometry insenstive quantity taking values in the interval $[0; 1]$, the value 0 being reached if and only if the object has a direct symmetry. DSI applies to finite set of points, with or without color constraints.

- χ, the chiral index of Petitjean (1997, 1999, 2002), which is a scale and isometry insensitive quantity taking values in the interval $[0; 1]$, the value 0 being reached if and only if the object has an indirect symmetry, i.e. meaning that the object is achiral. χ applies to discrete and continuous probability distributions, with or without color constraints. E.g., a finite set of n points is viewed as a sample of size n issued from a probability distribution, continuous or not. The chiral index measures a normalized distance between mirror images rather than the deviation from an idealized achiral object, because this latter is not ensured to exist: see Petitjean (2006).

Once defined asymmetry measures, the next problem which arises is to characterize the most asymmetrical objects, if existing. The calculation has been made for various classes of objects. We give in table 1 the examples of the most asymmetric triangles.

In the case the colors are discarded, the chiral index is an asymmetry coefficient applicable to probability distributions of random vectors taking values in the d-dimensional space $\mathbb{R}^d$, $d \geq 1$. When $d = 1$, at the difference of the widely used skewness introduced by Pearson (1895), the chiral index is null if and only if the distribution is achiral. As mentioned by Petitjean (1997) and in Petitjean (2003, section 2.9), the chiral index of a sample of n real values is easy to compute: sort the values in increasing order, then in decreasing order, and compute the correlation coefficient r_m between these two ordered samples: $\chi = (1 + r_m)/2$.

3.3 Seeking the most asymmetric objects

Let us come back to the problem of characterizing the most asymmetric distributions (in the sense of indirect symmetry, i.e. mirror symmetry). The simplest example is the one of three equiprobable points on the real line. As calculated in Petitjean (2013), $\chi = (1-\alpha)^2/4(1+\alpha+\alpha^2)$, where α is ratio of the lengths of the two adjacent segments defined by the three points. It can be shown that the most asymmetric distribution is reached when $\alpha = 0$, i.e. when two of the three points have the same abscissa, which in fact means that we have one point with a weight of 1/3 and one point with a weight of 2/3: the maximal chiral index is here equal to $1/4$. In this situation, we should speak about about the probability distribution of a random variable taking two values, one with a probability p=1/3 and the other one with a probability q=2/3 (thus p+q=1). Such a probability distribution is usually called a Bernoulli distribution with parameter $p = 1/3$ (see any textbook on statistics).

Going further in our quest of the most asymmetric distribution, it was shown by Petitjean (2002) that, for a random variable, the upper bound of χ is $1/2$, and it can be asymptotically approached for the Bernoulli distribution with the parameter p tending either to 0 or to 1.

Now let us look at the characterization of the maximally chiral (asymptotically or not) distributions in the plane when $d \geq 2$. The calculations are by for more difficult. When $d = 2$, Coppersmith and Petitjean (2005) show that the upper bound of χ fall on the interval $[1 - 1/\pi; 1 - 1/2\pi]$ and gave an awkward example of a family of distributions for which χ approaches asymptotically the limiting value $1 - 1/\pi$, which they conjectured to be optimal.

But this is still an open problem. In higher-dimensional spaces (i.e. $d \geq 3$), Petitjean (2008) shown that the upper bound of χ is somewhere in $[1/2; 1]$, but characterizing the most chiral distributions remains an open problem.

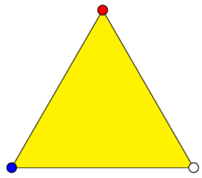
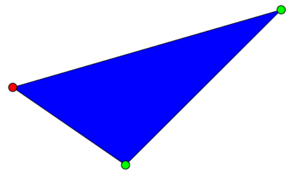
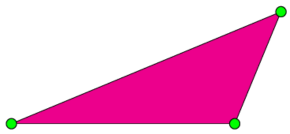
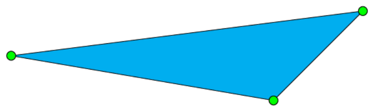
Number of equivalent vertices	The most asymmetric triangle (minimal direct symmety)	The most chiral triangle (minimal indirect symmetry)
None	Does not exist	 $\chi = 1$ Equilateral triangle
2	$DSI = 1$	 $\chi = 1 - \sqrt{2}/2 \approx 0.292893$
3	 $DSI = 1 - \sqrt{2}/2 \approx 0.292893$	 $\chi = 1 - 2\sqrt{5}/5 \approx 0.105573$

Table 1: The most asymmetric triangles, according to (Petitjean, 1997, 1999), in function of the number of equivalent vertices: two vertices are equivalent when they have the same color. The triangles are represented by their colored vertices. The colors filling the interior of the triangles have no meaning: they were added for clarity.
Top: the unequivalence of all vertices precludes the existence of any direct symmetry, while the the most chiral triangle with three unequivalent vertices (in yellow) is equilateral.
Mid: the most asymmetric triangle with two equivalent vertices is degenerate: its aligned vertices are at abscissas proportional to $(-1-\sqrt{3})/2$, $(-1+\sqrt{3})/2$, 1; the most chiral triangle with two equivalent vertices (in dark blue) has sidelengths ratios $\sqrt{1-\sqrt{6}/4} : 1 : \sqrt{1+\sqrt{6}/4}$.
Bottom: the most asymmetric triangle with three equivalent vertices (in light red) has angles: $\pi/4$, $\pi/8$, $5\pi/8$; the most chiral triangle with three equivalent vertices (in light blue) has sidelengths ratios $1 : \sqrt{4+\sqrt{15}} : \sqrt{(5+\sqrt{15})/2}$.

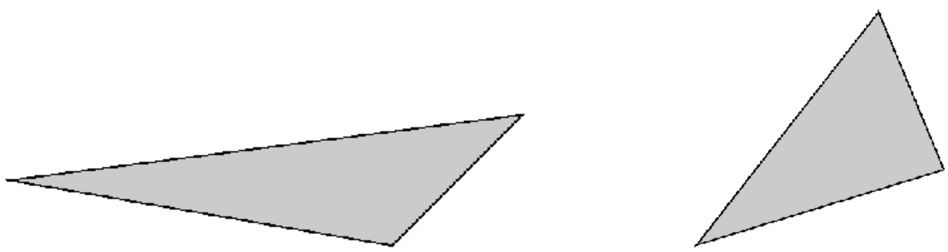


Figure 2: Comparison of the two *most scalene* triangles. On the left: the obtuse triangle presented at the right bottom of table 1. On the right: the triangular face of the most chiral disphenoid defined by Petitjean (2015); the coordinates are given in annex C; the sidelengths ratios are $1 : \sqrt{3-\sqrt{2}/2} : \sqrt{3}$. Strangely, squaring these ratios defines the sidelengths of an obtuse triangle shown to be the least symmetric one in the sense of Bowden *et al* (2018, see figure 7).

However, in the three-dimensional space the maximum of the chiral index can be computed for specifice classes of objects. An example is the most chiral disphenoid, for which the chiral index $\chi = 3(13 - 6\sqrt{2})/97 \approx 0.139630$: see Petitjean (2015). The triangular faces of this disphenoid are congruent to a scalene triangle which can be used for a funny application: anybody can remember how it was difficult for the teacher to draw a scalene triangle on the blackboard because it may be either too close to a right triangle or too close to an isosceles triangle; thus we propose to retain the latter scalene triangle because, as a face of the most chiral disphenoid, it is the farthest possible to the right triangle and to the isosceles triangle which both induce an achiral disphenoid. Of course, if an obtuse triangle is needed, no disphenoid can be built and thus we rather recommend the most chiral triangle in light blue presented at the right bottom of table 1. Both triangles are displayed in figure 2. Other *most scalene* triangles were proposed by Robin (2009), and by Bowden *et al* (2018), but their builduing process cannot be generalized to other least symmetric figures.

Other examples of chiral three-dimensional objects are molecules: see IUPAC (1996). A commonly used molecular model is such that the atoms are represented by the nodes of a graph and the chemical bonds are represented by the edges connecting the atoms. This graph is realized in the three-dimensional space by attributing measured or computed coordinates to the atoms, considered to be ponctual. Thus the chiral index can be computed for any molecular conformer. Looking for extreme chirality molecules, Alan Schwartz designed the chiralanes and chirolanes families of molecules: see Schwartz (2004a). The most remarkable chiralane, shown in figure 3, has a chiral index close to 1.

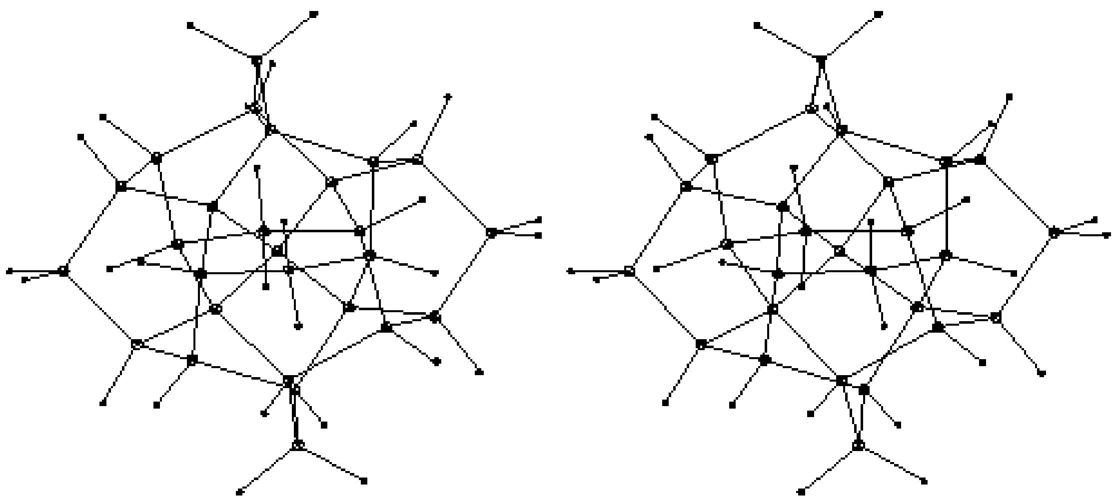


Figure 3: The two mirror images of the [6.6]chiralane, designed by Alan Schwartz. The chiral index of this compound is $\chi = 0.9824$ and the one of its carbon skeleton is $\chi = 1.000$; image and data from Schwartz and Petitjean (2008).

Seeking extreme chirality molecules has a potential application in physics: Schwartz (2004b, 2008) suggested that the Equivalence Principle (i.e. all substances fall at equal speeds independently of composition and internal structure) could be falsified for enantiomeric pairs.

IV DISCUSSION AND CONCLUSION

Looking for the least symmetric figures or distributions is a topic neglected in the literature. It has been suggested by Field and Golubitsky (2009) that chaos could be the antithesis of symmetry because chaos is regarded to be chracterized by unpredictability and complexity, and because chaos is defined in the dictionary as ”any condition or place of total disorder or confusion”. This is unclear for the mathematician: in turn disorder needs to be defined, and then how to define the maximal disorder?

Then, it is sometimes assumed that an other way to look at the opposite of symmetry could be to operate at random. Consider a rectangular domain of the plane and generate a sequence of samples of increasing sizes following the uniform law over this rectangular domain. In virtue of a theorem of Petitjean (2002), the sequence of chiral indices of these samples converges to the chiral index of the parent population, which is zero: in this situation, ”more random there is, more symmetry there is”. So, which sample issued from this parent population could be expected to have the highest chiral index?

Looking at figure 1 and to its explanation given at the end of section II, we understand now why the location of the glasses should have been neither chaotic nor at random: none of these choices would have been pertinent to evoke asymmetry as best as possible. What took to the scientist several years of hard mathematical work was immediate evidence for the artist. Following the intuition of Caroline Bouissou, we think that there is no perfectly asymmetric object or figure (in the sense of mirror symmetry), without colors. Thus we emit the conjecture that the upper bound of the chiral index will never reach the value 1 for an unit mass distribution in the d-dimensional space, $d \geq 3$ (i.e. for a probability distribution of a random vector), because we think that no chiral distribution can approach the limiting value 1, even asymptotically. This latter is reachable only when color constraints are added (see section 3.2). Here again the artist was right: the glasses of figure 1 have no color, they are transparent!

This shows why a collaboration with artists can be enlighting for the scientists. Such a collaboration is an asymmetric exchange: the scientist takes benefits from the creativity of the artist while the artist takes benefits from the mathematical results proved by the scientist.

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A ANNEX 1

There are many journals and newsletters where symmetry papers dealing with arts and mathematics or sciences can be read. We report in table 2 some examples of such journals and newsletters. There are many other ones but we warn the readers that several of them are classified as predatory: see Beall (2013, 2016); Sorokowski *et al* (2017).

Journal, newsletter or proceedings	ISSN	Publisher
ACM Transactions on Graphics (TOG)[1]	0730-0301	ACM
Bridges annual proceedings[2]	-	Bridges Organization[3]
Computers & Mathematics with Applications[4]	0898-1221	Elsevier
ESMA Newsletter[5]	-	ESMA
Hyperseeing[6]	-	ISAMA
Journal of Mathematics and the Arts[7]	1751-3480	Taylor & Francis
Leonardo[8]	0024-094X	MIT Press
Nexus Network Journal[9]	1522-4600	Springer
Symmetry[10]	2073-8994	MDPI
Symmetry: Art and Science[11]	1447-607X	ISIS-Symmetry[12]
Symmetry: Culture and Science[13]	0865-4824	Symmetrion
VisMath[14]	1821-1437	ISIS-Symmetry

Table 2: Some journals or newsletters containing papers dealing with symmetry and arts.

[1]Computer sciences oriented; publishes the proceedings of the SIGGRAPH conferences (there is a digital arts section): see SIGGRAPH (1974).

[2]At the interface art-mathematics; archives: see Bridges (1998).

[3]Founded by Reza Saranghi: see Sarhangi (1998), Fenyvesi (2016), Shrestha (2016).

[4]Several papers related to arts were grouped in a special issue: see Hargittai (1986).

[5]See ESMA (2010).

[6]Newsletter of ISAMA (at least one issue per year from 2006 to 2014): see ISAMA (1998), Friedman and Akleman (2006).

[7]See Greenfield (2007).

[8]Journal of the International Society for the Arts, Sciences and Technology (ISAST): see Malina (1968).

[9]Related to architecture and thus related to arts; see Williams (1999).

[10]Publishes sometimes special issues related to arts: see Sasaki (2010), Tyler (2011).

[11]Founded in 2001 by Nagy and Lugosi (2001) and presented as the continuation of "Symmetry: Culture and Science" while the latter is still published; has published the proceedings of several symmetry conferences organized by ISIS-Symmetry since 2001.

[12]It was suggested to use SIS as an alternative abbreviation of the name of the Society: see ISIS (2016).

[13]The oldest journal fully devoted to all aspects of symmetry, founded by Darvas and Nagy (1990); includes issues or sections related to arts, science, mathematics, literature, education, etc.; since 2003 the board is the one of the International Symmetry Association: see ISA (2003); has published the proceedings of symmetry conferences organized by ISIS-Symmetry and then the proceedings of Symmetry Festival conferences organized by the ISA.

[14]E-journal, founded in 1999 by Slavik Jablan; published in 2014 by the Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade: see Radović (2014); no issue posterior to 2014 is mentioned.

B ANNEX 2

Symmetry and Asymmetry artworks below were all realized by Caroline Bouissou.

Figure 4 *Orchids* and 5 *Home war* deal with left and right chirality introducing a human paradox of world understanding.

Figure 6 *Ambiguous landscape* and 7 *Observatoire* show a landscape work dealing with a fragmentation of composition where symmetry and assymetry play as a dynamic.

Figure 4: *Orchids 2017: drawing serie by Caroline Bouissou.* Photograph: © Caroline Bouissou. Drawings with red pen realized with the left hand and drawings wih the blue pen realized with the right hand. Two sketchbooks. 200 drawings.

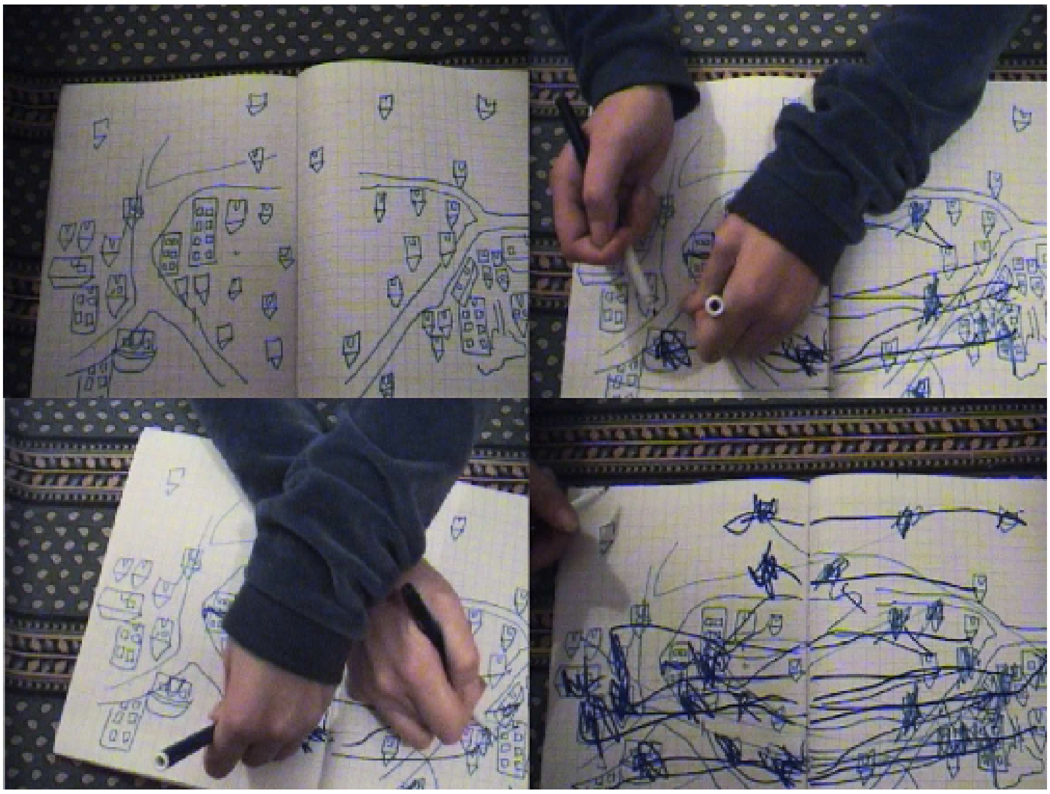

Figure 5: *Home war 2004: video by Caroline Bouissou.* Photograph: © Caroline Bouissou. Bilateral war committing only a single body and building a drawing. On the kitchen table, the ongoing war creates a drawing. Each adversary belongs to the same body. The drawing constructs itself, but what is more important is the process.

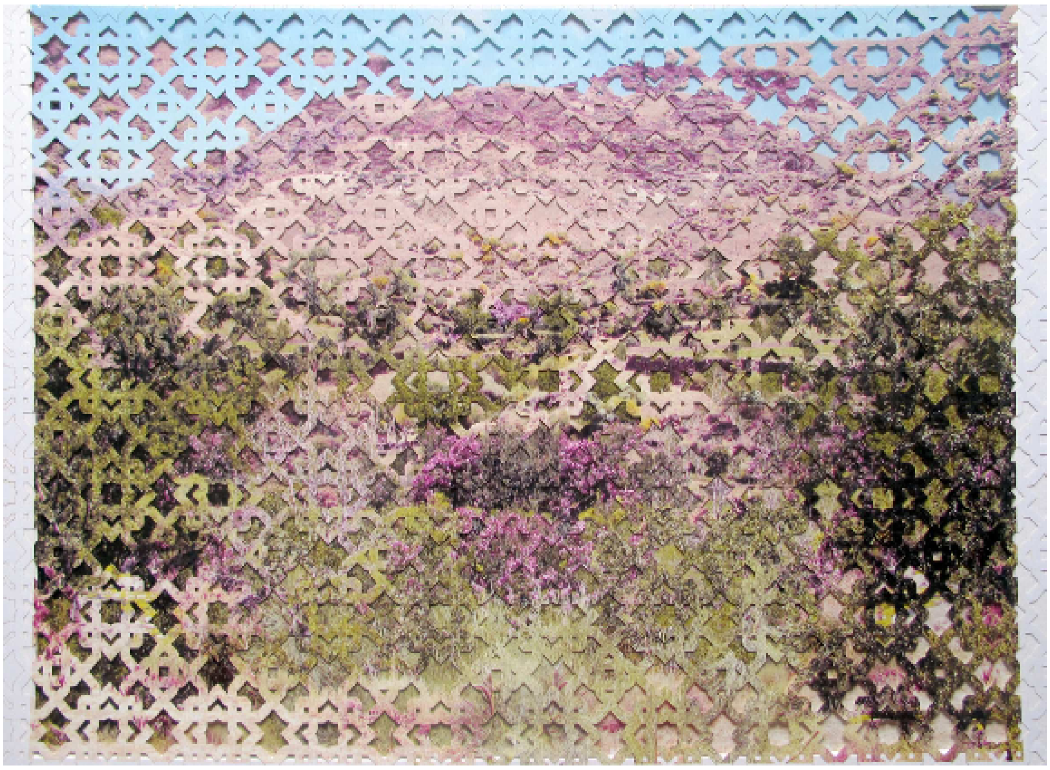

Figure 6: *Ambiguous landscapes 2015-...* Photographs by © Caroline Bouissou. Photographs layers cut onto geometrical patterns (2015–ongoing). The addition of patterns layers creates a complex image.

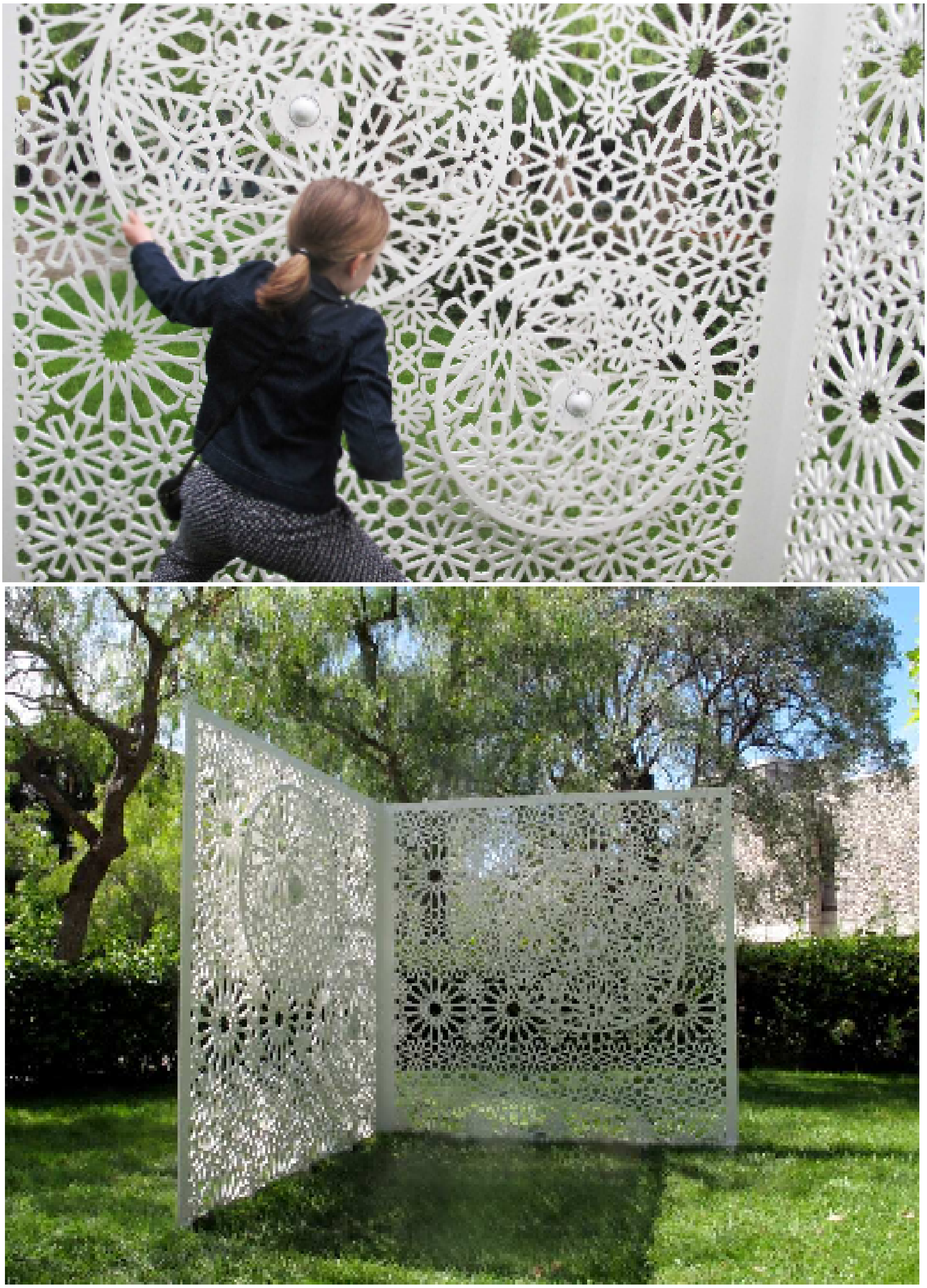

Figure 7: *Observatoire 2015* (continued on next page): kinetic sculpture by Caroline Bouissou. Photograph: © Caroline Bouissou.

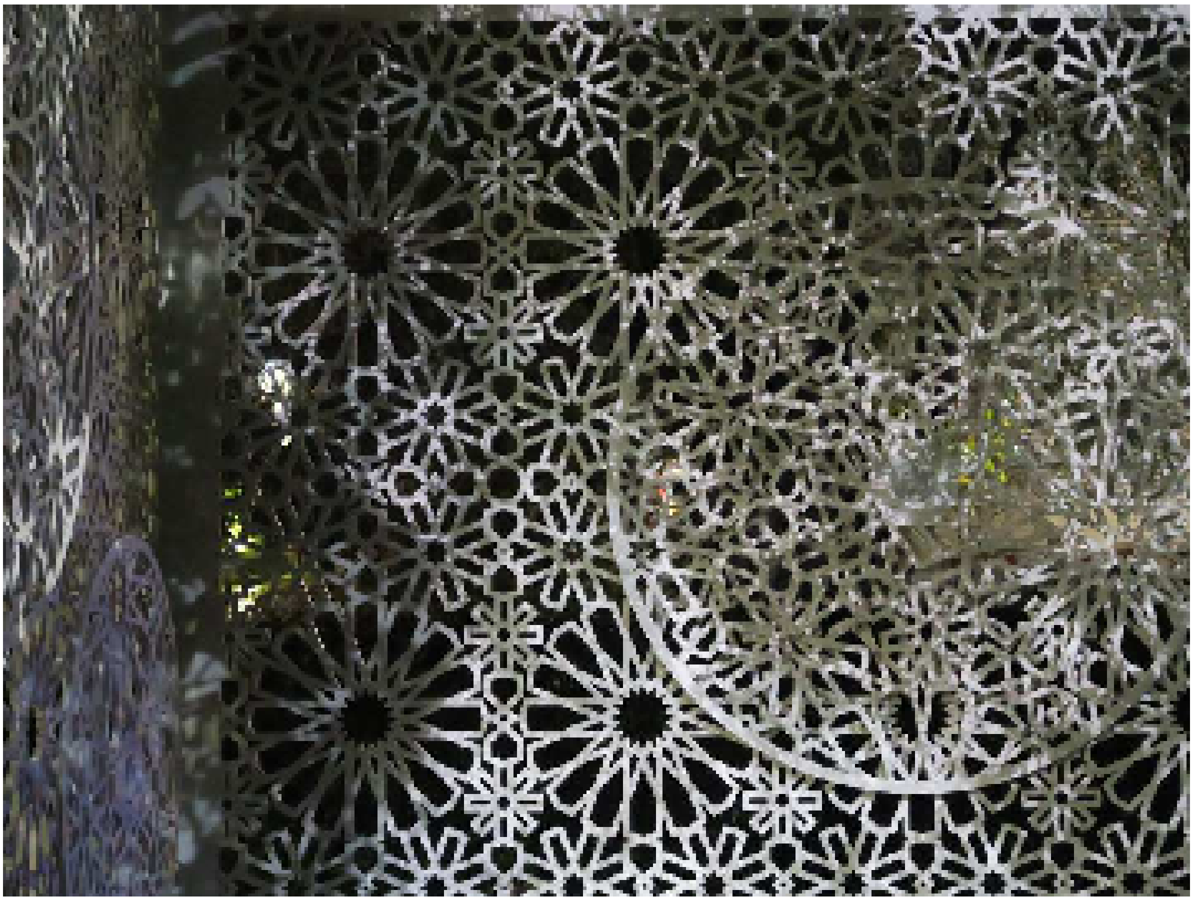

Figure 7: continued. *Observatoire 2015*: kinetic sculpture by Caroline Bouissou. Photograph: © Caroline Bouissou. 78.74 × 157.48 × 0.59 inches submarine iron sheet folded and cut where are fixed three activatable wheels. The pattern composed by geometrical forms (geometricization of natural shapes) plays with the garden we see thought. Wheels movements modulate the viewer perception when patterns compositions overlap.

C ANNEX 3

The most scalene triangle of Petitjean (2015) is congruent to the one of which the coordinates of the vertices p_1, p_2 and p_3 are given below. These coordinates satisfy to: $p_1 + p_2 + p_3 = 0$.

We define $\omega = \left[\sqrt{2 + 36(3 + 2\sqrt{2})} - (6 + 5\sqrt{2})\right]/12$; $\quad (\omega \approx 0.123590)$.

$$\boxed{p_1 \mid p_2 \mid p_3} \; = \; \begin{array}{|c|c|c|} \hline 19 + 8\sqrt{2} + \omega(-25 + 34\sqrt{2}) & -8 - 4\sqrt{2} + \omega(11 - 20\sqrt{2}) & -11 - 4\sqrt{2} + \omega(14 - 14\sqrt{2}) \\ -2 - 4\sqrt{2} + \omega(1 - 11\sqrt{2}) & 13 + 5\sqrt{2} + \omega(1 + 7\sqrt{2}) & -11 - \sqrt{2} + \omega(-2 + 4\sqrt{2}) \\ \hline \end{array}$$

$$\boxed{p_1 \mid p_2 \mid p_3} \; \approx \; \begin{array}{|r|r|r|} \hline 33.166579 & -15.793025 & -17.373554 \\ -9.455880 & 21.418141 & -11.962261 \\ \hline \end{array}$$

The angles at these vertices are respectively about 35.075, 60.475 and 84.450 degrees.

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